Galois groups of the Lie-irreducible generalized q-hypergeometric equations of order three with q-real parameters : an approach using a density theorem

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Abstract. *In this paper we compute the difference Galois groups of the Lie-irreducible regular singular generalized q-hypergeometric equations of order 3 with q-real parameters by using a density theorem due to Sauloy*[1]*. In contrast with the differential case, we show that these groups automatically contain the special linear group* $Sl_3(\mathbb{C})$.

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[1]An alternative approach would be to use duality. This is a work in progress and will appear elsewhere.

In the whole paper q is a complex number such that $0 < |q| < 1$.

1 Generalized hypergeometric series and equations and q-analogues

1.1 The differential case

In accordance with the tradition, we denote by δ the Euler differential operator : $\delta = z\frac{d}{dz}$. Consider $(r,s) \in \mathbb{N}^{*2}$, $\underline{\alpha} = (\alpha_1, ..., \alpha_r) \in \mathbb{C}^r$ and $\underline{\beta} = (\beta_1, ..., \beta_s) \in \mathbb{C}^s$. Recall that the *generalized hypergeometric series* with parameters $(\underline{\alpha}, \underline{\beta})$ is given by :

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \frac{(\underline{\alpha})_n}{(\underline{\beta})_n} z^n \\
=\ &\sum_{n=0}^{+\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} z^n \\
=\ &\sum_{n=0}^{+\infty} \frac{\alpha_1(\alpha_1+1)\cdots(\alpha_1+n-1)\cdots\alpha_r(\alpha_r+1)\cdots(\alpha_r+n-1)}{\beta_1(\beta_1+1)\cdots(\beta_1+n-1)\cdots\beta_s(\beta_s+1)\cdots(\beta_s+n-1)} z^n
\end{aligned}$$

and that it provides us with a solution of the following linear differential equation :

$$\left(\prod_{j=1}^{s} (\delta + \beta_j - 1) - z \prod_{i=1}^{r} (\delta + \alpha_i) \right) y = 0 \tag{1}$$

called the *generalized hypergeometric equation* with parameters $(\underline{\alpha}, \underline{\beta})$.

These series and equations have retained the attention of many authors. Since the pioneering work of Gauss, the theory of generalized hypergeometric series and equations has figured importantly in many branches of mathematics, from mathematical physics to arithmetic. We shall now discuss more specifically the Galois theory of hypergeometric equations.

As regards functional equations, the study of (1) leads us to distinguish two cases :

- when $r = s$ the equation (1) is Fuchsian over the Riemann sphere with three singular points : 0, 1 and ∞;

- whereas when $r \neq s$ the equation (1) has two singular points : 0 and ∞, one of these points being an *irregular singularity* (the other one is regular).

The Galois theory of the generalized hypergeometric equations in both regular and irregular cases was in particular studied in details by Beukers, Brownawell and Heckman in [3, 2] and by Katz in [11]. In [7, 12] A. Duval and C. Mitschi computed the Galois groups of some irregular generalized hypergeometric equations by means of Ramis density theorem.

From a technical point of view, in the Fuchsian case ($r = s$), the determination of the Galois groups of the *Lie-irreducible* generalized hypergeometric equations (which are, roughly speaking, equations with big Galois groups) rely on the fact that the local monodromy of (1)

around the singular point $z = 1$ is a pseudo-reflection : this allows us to apply a general result on algebraic groups (see [3, 11]).

The present paper is concerned with the Galois theory of the natural counterpart for q-difference equations of the generalized hypergeometric equations with $r = s = 3$.

1.2 The q-difference case

The natural q-analogues of the generalized hypergeometric equations and series are respectively called the generalized q-hypergeometric equations and series. Recall that the *generalized q-hypergeometric series* with parameters $(\underline{a}, \underline{b}) \in (\mathbb{C}^*)^r \times (\mathbb{C}^*)^s$ is defined by :

$$\sum_{n=0}^{+\infty} \frac{(\underline{a}; q)_n}{(\underline{b}; q)_n} z^n$$

and that, similarly to the differential case, it satisfies a functional equation, namely the *generalized q-hypergeometric equation* with parameters $(\underline{a}, \underline{b})$, denoted by $\mathcal{H}_q(\underline{a}; \underline{b})$, and given by :

$$\left(\prod_{j=1}^{s} (\frac{b_j}{q}\sigma_q - 1) - z \prod_{i=1}^{r} (a_i \sigma_q - 1) \right) \phi(z) = 0 \tag{2}$$

where σ_q denotes the scaling operator acting on a function ϕ by $(\sigma_q \phi)(z) = \phi(qz)$. We have used the classical notations for q-Pochhammer symbols :

$$(a_i; q)_n = (1 - a_i)(1 - a_i q) \cdots (1 - a_i q^{n-1})$$

and :

$$(\underline{a}; q)_n = (a_1; q)_n \cdots (a_r; q)_n$$

(similar notations for $\underline{b}$). It is usual to normalize both these equations and these series by requiring $b_1 = q$: we will implicitly do this hypothesis in the whole paper. The above q-hypergeometric series is then denoted by ${}_r\phi_{s-1}(\underline{a}; \underline{b}; z)$.

As in the differential case, one can easily check that the q-difference equation $\mathcal{H}_q(\underline{a}; \underline{b})$ is Fuchsian (see section 2.1 for this notion) if and only if $r = s$.

In this paper, we focus our attention on the case $r = s = 3$. The functional equation (1) is then equivalent to a 3×3 functional system. Indeed, with the following notations :

$$\lambda(\underline{a}; \underline{b}; z) = \frac{1 - z}{b_2 b_3 / q^2 - z a_1 a_2 a_3}, \qquad \mu(\underline{a}; \underline{b}; z) = \frac{z(a_1 + a_2 + a_3) - (1 + b_2/q + b_3/q)}{b_2 b_3 / q^2 - z a_1 a_2 a_3}$$

and :

$$\delta(\underline{a}; \underline{b}; z) = \frac{(b_2 b_3 / q^2 + b_2/q + b_3/q) - z(a_1 a_2 + a_2 a_3 + a_1 a_3)}{b_2 b_3 / q^2 - z a_1 a_2 a_3}$$

a function ϕ is solution of the equation (2) if and only if the vector $\Phi(z) = \begin{pmatrix} \phi(z) \\ \phi(qz) \\ \phi(q^2 z) \end{pmatrix}$ satisfies the following functional system :

$$\Phi(qz) = A(\underline{a}; \underline{b}; z)\Phi(z) \tag{3}$$

with :

$$A(\underline{a};\underline{b};z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda(\underline{a};\underline{b};z) & \mu(\underline{a};\underline{b};z) & \delta(\underline{a};\underline{b};z) \end{pmatrix}.$$

Our main purpose is to show that the Galois groups of any Lie-irreducible regular singular generalized q-hypergeometric equation of order 3 with q-real parameters (that is of the form q^α with $\alpha \in \mathbb{R}$) necessarily contains the special linear group. We emphasize that the similar statement for differential hypergeometric equations is *false* (see [11]).

A number of authors have developed q-difference Galois theories over the past years, among whom Franke [9], Etingof [8], Van der Put and Singer [19], Van der Put and Reversat [20], Chatzidakis and Hrushovski [5], Sauloy [15], André [1], etc. The exact relations between the existing Galois theories for q-difference equations are partially understood. For this question, we refer the reader to [4], and also to our Remark section 2.2.

We will deal in this paper with the (Tannakian) Galois groups in the sense of Sauloy in [15] (see section 2 for an overview of the theory).

The notion of *Lie-irreducibility* used in this paper is a natural extension to q-difference equations of a notion introduced by Katz : an equation is Lie-irreducible if the linear representation of the neutral component of its Galois group induced by the linear representation of the Galois group itself given by the Tannakian duality is irreducible.

From a technical point of view, the fundamental fact which deeply distinguishes the q-difference case from the differential case is that we do not have presently a notion of local monodromy around the singular point $z = 1$ (or $q^{\mathbb{Z}}$); nevertheless, we have some avatars built up from Birkhoff connection matrices (generators of the connection component : see section 2.2). Whatever, it seems that we cannot use the *pseudo-reflection trick* mentioned above in order to determine the Galois groups of the Lie-irreducible equations. This is an important problem. Our method, in the three dimensional case, is based on :

- the classification of unimodular semi-simple algebraic subgroups of $\mathsf{Sl}_3(\mathbb{C})$ acting irreducibly on $\mathbb{C}^3$;
- a q-analogue of Schlesinger density theorem stated and established by Sauloy in [15].

Last, we would like to point out André's paper [1] which contains a nice computation of the Galois groups of some generalized q-hypergeometric equations using a *specialization* procedure.

1.3 Organization - Main result

The paper is organized as follows. In section 2, we give a brief overview of the Galois theory of Fuchsian q-difference systems in the sense of Sauloy. In section 3 we introduce the notions of *irreducibility*, of *Lie-irreducibility* and we discuss the irreducibility of the generalized q-hypergeometric equations. In section 4, we investigate the properties of some algebraic subgroups of $\mathsf{Gl}_3(\mathbb{C})$. Section 5 is devoted to the proof of our main theorem :

Theorem 1. *Let* G *be the Galois group of a Lie-irreducible generalized q-hypergeometric equation* $\mathcal{H}_q(\underline{a};\underline{b})$ *of order three with q-real parameters. Then* $\mathsf{G}^{0,\mathsf{der}} = \mathsf{Sl}_3(\mathbb{C})$*. More precisely :*

- $\mathsf{G} = \mathsf{Gl}_3(\mathbb{C})$ *if* $\frac{a_1a_2a_3}{b_2b_3} \notin q^{\mathbb{Z}}$*;*
- $\mathsf{G} = \overline{\langle \mathsf{Sl}_3(\mathbb{C}), e^{2\pi i(\beta_2+\beta_3)}, v_1v_2\rangle}$ *if* $\frac{a_1a_2a_3}{b_2b_3} \in q^{\mathbb{Z}}$*.*

As usually, $\mathsf{G}^{0,\mathsf{der}}$ denotes the derived group of G^0, the neutral component of G.

2 Galois theory for regular singular q-difference equations

Using analytic tools together with Tannakian duality, Sauloy developed in [15] a Galois theory for regular singular q-difference systems. In this section, we shall first recall the principal notions used in [15], mainly the Birkhoff matrix and the twisted Birkhoff matrix. Then we shall explain briefly that this lead to a Galois theory for regular singular q-difference systems. Last, we shall state a density theorem for these Galois groups, which will be of main importance in the rest of the paper.

2.1 Basic notions

Let us consider $A \in Gl_n(\mathbb{C}(\{z\}))$. Following Sauloy in [15] (see also [14]), the q-difference system :

$$Y(qz) = A(z)Y(z) \tag{4}$$

is said to be *Fuchsian* at 0 if A is holomorphic at 0 and if $A(0) \in \mathsf{Gl}_n(\mathbb{C})$. Such a system is *non-resonant* at 0 if, in addition, $Sp(A(0)) \cap q^{\mathbb{Z}^*}Sp(A(0)) = \emptyset$. Last we say that the above q-difference system is *regular singular* at 0 if there exists $R^{(0)} \in \mathsf{Gl}_n(\mathbb{C}(\{z\}))$ such that the q-difference system defined by $(R^{(0)}(qz))^{-1}A(z)R^{(0)}(z)$ is Fuchsian at 0. We have similar notions at ∞ using the variable change $z \leftarrow 1/z$.

In case that the system is global, that is $A \in \mathsf{Gl}_n(\mathbb{C}(z))$, we say that the system (4) is *Fuchsian* (resp. *Fuchsian and non-resonnant*, *regular singular*) if it is Fuchsian (resp. Fuchsian and non-resonnant, regular singular) both at 0 and at ∞.

For instance, the basic hypergeometric system (3) is Fuchsian.

Local solution at 0. Suppose that (4) is Fuchsian and non-resonant at 0 and consider $J^{(0)}$ a Jordan normal form of $A(0)$. Thanks to [15] there exists $F^{(0)} \in \mathsf{Gl}_n(\mathbb{C}\{z\})$ such that :

$$F^{(0)}(qz)J^{(0)} = A(z)F^{(0)}(z). \tag{5}$$

Therefore, if $e^{(0)}_{J^{(0)}}$ denotes a fundamental system of solutions of the q-difference system with constant coefficients $X(qz) = J^{(0)}X(z)$, the matrix-valued function $Y^{(0)} = F^{(0)}e^{(0)}_{J^{(0)}}$ is a fundamental system of solutions of (4). In [15], $e^{(0)}_{J^{(0)}}$ is defined as follows. We denote by θ_q the Jacobi theta function defined by $\theta_q(z) = (q;q)_\infty\,(z;q)_\infty\,(q/z;q)_\infty$. This is a meromorphic function over $\mathbb{C}^*$ whose zeros are simple and located on the discrete logarithmic spiral $q^{\mathbb{Z}}$. Moreover, the functional equation $\theta_q(qz) = -z^{-1}\theta_q(z)$ holds. Now we introduce, for all $\lambda \in \mathbb{C}^*$

such that $|q| \leq |\lambda| < 1$, the q-character $e_\lambda^{(0)} = \frac{\theta_q}{\theta_{q,\lambda}}$ with $\theta_{q,\lambda}(z) = \theta_q(\lambda z)$ and we extend this definition to an arbitrary nonzero complex number $\lambda \in \mathbb{C}^*$ requiring the equality $e_{q\lambda}^{(0)} = z e_\lambda^{(0)}$. If $D = P\mathrm{diag}(\lambda_1, ..., \lambda_n)P^{-1}$ is a semisimple matrix then we set $e_D^{(0)} := P\mathrm{diag}(e_{\lambda_1}^{(0)}, ..., e_{\lambda_n}^{(0)})P^{-1}$. It is easily seen that it does not depend on the chosen diagonalization. Furthermore, consider $\ell_q(z) = -z\frac{\theta_q'(z)}{\theta_q(z)}$ and, if U is a unipotent matrix, $e_U^{(0)} = \Sigma_{k=0}^n \ell_q^{(k)}(U - I_n)^k$ with $\ell_q^{(k)} = \binom{\ell_q}{k}$. Finally if $J^{(0)} = D^{(0)}U^{(0)}$ is the multiplicative Dunford decomposition of $J^{(0)}$, with $D^{(0)}$ semi-simple and $U^{(0)}$ unipotent, we set $e_{J^{(0)}}^{(0)} = e_{D^{(0)}}^{(0)} e_{U^{(0)}}^{(0)}$.

Local solution at ∞. Using the variable change $z \leftarrow 1/z$, we have a similar construction at ∞. The corresponding fundamental system of solutions is denoted by $Y^{(\infty)} = F^{(\infty)} e_{J^{(\infty)}}^{(\infty)}$.

Throughout this section we assume that (4) has coefficients in $\mathbb{C}(z)$ and that it is Fuchsian and non-resonant.

Birkhoff matrix. The linear relations between the two fundamental systems of solutions introduced above are given by the Birkhoff matrix (also called connection matrix) $P = (Y^{(\infty)})^{-1} Y^{(0)}$. Its entries are elliptic functions *i.e.* meromorphic functions over the elliptic curve $\mathbb{E}_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

Twisted Birkhoff matrix. In order to describe a Zariki-dense set of generators of the Galois group associated to the system (4), we introduce a "twisted" connection matrix. According to [15], we choose for all $z \in \mathbb{C}^*$ a group endomorphism g_z of $\mathbb{C}^*$ sending q over z. Before giving an explicit example, we have to introduce some notations. Let us consider $\tau \in \mathbb{C}$ such that $q = e^{-2\pi i \tau}$ and set, for all $y \in \mathbb{C}$, $q^y = e^{-2\pi i \tau y}$. We also define the (non continuous) function $\log_q$ on the whole punctured complex plane $\mathbb{C}^*$ by $\log_q(q^y) = y$ if $y \in \mathbb{C}^* \setminus q^{\mathbb{R}}$ and we require that its discontinuity is just before the cut when turning counterclockwise around 0. We can now give an explicit example of endomorphism g_z namely the function $g_z : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \to \mathbb{C}^*$ sending uq^ω to $g_z(uq^\omega) = z^\omega = e^{-2\pi i \tau \log_q(z)\omega}$ for $(u, \omega) \in \mathbb{U} \times \mathbb{R}$, where $\mathbb{U} \subset \mathbb{C}$ is the unit circle.

Then we set, for all z in $\mathbb{C}^*$, $\psi_z^{(0)}(\lambda) = \frac{e_{q,\lambda}(z)}{g_z(\lambda)}$ and we define $\psi_z^{(0)}\left(D^{(0)}\right)$, the *twisted factor* at 0, by $\psi_z^{(0)}\left(D^{(0)}\right) = P\mathrm{diag}(\psi_z^{(0)}(\lambda_1), ..., \psi_z^{(0)}(\lambda_n))P^{-1}$ with $D^{(0)} = P\mathrm{diag}(\lambda_1, ..., \lambda_n)P^{-1}$. We have a similar construction at ∞ by using the variable change $z \leftarrow 1/z$. The corresponding twisting factor is denoted by $\psi_z^{(\infty)}(J^{(\infty)})$.

Finally, the twisted connection matrix $\breve{P}(z)$ is :

$$\breve{P}(z) \quad = \quad \psi_z^{(\infty)}\left(D^{(\infty)}\right) P(z) \psi_z^{(0)}\left(D^{(0)}\right)^{-1}.$$

2.2 Definition of the difference Galois groups

The definition of the Galois groups of regular singular q-difference systems given by Sauloy in [15] is somewhat technical and long. Here we do no more than describe the underlying idea.

(Global) Galois group. Let us denote by $(\mathcal{E}, \otimes)$ the rigid abelian tensor category of regular singular q-difference systems with coefficients in $\mathbb{C}(z)$. This is actually a Tannakian category

over $\mathbb{C}$ but the existence of a fiber functor is not obvious. We shall now explain how one can get a complex valued fiber functor *via* a Riemann-Hilbert correspondance.

The category $\mathcal{E}$ is equivalent to the category $\mathcal{C}$ of connection triples whose objects are triples $(A^{(0)}, P, A^{(\infty)}) \in \mathsf{Gl}_n(\mathbb{C}) \times \mathsf{Gl}_n(\mathcal{M}(\mathbb{E}_q)) \times \mathsf{Gl}_n(\mathbb{C})$ (we refer to [15] for the complete definition of $\mathcal{C}$). Furthermore $\mathcal{C}$ can be endowed with a tensor product $\underline{\otimes}$ making the above equivalence of categories compatible with the tensor products. Let us emphasize that $\underline{\otimes}$ is not the usual tensor product for matrices. Indeed some twisting factors appear because of the bad multiplicative properties of the q-characters $e_{q,c}$: in general $e_{q,c}e_{q,d} \neq e_{q,cd}$.

The category rigid abelian tensor category $\mathcal{C}$ is a Tannakian category over $\mathbb{C}$. The functor ω_0 from $\mathcal{C}$ to $Vect_{\mathbb{C}}$ sending an object $(A^{(0)}, P, A^{(\infty)})$ to the underlying vector space $\mathbb{C}^n$ on which $A^{(0)}$ acts is a fiber functor. Let us remark that there is a similar fiber functor ω_∞ at ∞. Following the general formalism of the theory of Tannakian categories (see [6]), the *absolute Galois group* of $\mathcal{C}$ (or, using the above equivalence of categories, of $\mathcal{E}$) is defined as the pro-algebraic group $\mathsf{Aut}^{\underline{\otimes}}(\omega_0)$ and the *global Galois group of an object* χ of $\mathcal{C}$ (or, using the above equivalence of categories, of an object of $\mathcal{E}$) is the complex linear algebraic group $\mathsf{Aut}^{\underline{\otimes}}(\omega_{0|\langle\chi\rangle})$ where $\langle\chi\rangle$ denotes the Tannakian subcategory of $\mathcal{C}$ generated by χ. For the sake of simplicity, we will often call $\mathsf{Aut}^{\underline{\otimes}}(\omega_{0|\langle\chi\rangle})$ the *Galois group* of χ (or, using the above equivalence of categories, of the corresponding object of $\mathcal{E}$).

Local Galois groups. Notions of local Galois groups at 0 and at ∞ are also available. They are subgroups of the (global) Galois group. Nevertheless, since these groups are of second importance in what follows, we omit the details and we refer the interesting reader to [15].

Remark. In [19], Van der Put and Singer showed that the Galois groups defined using a Picard-Vessiot theory can be recover by means of Tannakian duality : it is the group of tensor automorphisms of some complex valued fiber functor over $\mathcal{E}$. Since two complex valued fiber functors on a same Tannakian category are isomorphic, we deduce that Sauloy's and Van der Put and Singer's theories coincide.

In the rest of this section we exhibit some natural elements of the Galois group of a given Fuchsian q-difference system and we state a density theorem.

2.3 The density theorem

In what follows, we assume that the q-difference system (4) has coefficients in $\mathbb{C}(z)$ and that it is Fuchsian and non-resonant.

Fix a "base point" $y_0 \in \Omega = \mathbb{C}^* \setminus \{\text{zeros of } \det(P(z)) \text{ or poles of } P(z)\}$. Sauloy exhibits in [15] the following elements of the (global) Galois group associated to the q-difference system (4) :

1.a. $\gamma_1(D^{(0)})$ and $\gamma_2(D^{(0)})$ where :

$$\gamma_1 : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \to \mathbb{U}$$

is the projection over the first factor and :

$$\gamma_2 : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \to \mathbb{C}^*$$

is defined by $\gamma_2(uq^\omega) = e^{2\pi i \omega}$.

1.b. $U^{(0)}$.

2.a. $\breve{P}(y_0)^{-1}\gamma_1(D^{(\infty)})\breve{P}(y_0)$ and $\breve{P}(y_0)^{-1}\gamma_2(D^{(\infty)})\breve{P}(y_0)$.

2.b. $\breve{P}(y_0)^{-1}U^{(\infty)}\breve{P}(y_0)$.

3. $\breve{P}(y_0)^{-1}\breve{P}(z)$, $z \in \Omega$.

The following result is due to Sauloy [15].

Theorem 2. *The algebraic group generated by the matrices 1.a. to 3. is the (global) Galois group G of the q-difference system (4). The algebraic group generated by the matrices 1.a. and 1.b. is the local Galois group at 0 of the q-difference system (4). The algebraic group generated by the matrices 2.a. and 2.b. is the local Galois group at ∞ of the q-difference system (4).*

The algebraic group generated by the matrices involved in 3. is called the *connection component* of the Galois group G.

3 Irreducibility and Lie-irreducibility

Let us introduce some terminologies.

Let $\mathcal{S}$ be an object of $\mathcal{E}$. The fiber functor ω_0 induces a equivalence of tensor categories :

$$\langle \mathcal{S} \rangle \xrightarrow{\sim} (\text{finite dimensional } \mathbb{C}\text{-representations of } \mathsf{G} = \mathsf{Aut}^{\underline{\otimes}}(\omega_{0|\langle \mathcal{S} \rangle})).$$

A regular singular q-difference system $\mathcal{S}$ of order n is *irreducible* if it corresponds to an irreducible representation of its Galois group G; it is *Lie-irreducible* if the restriction to G^0 of the representation of G corresponding to $\mathcal{S}$ is irreducible.

We have the following obvious implication :

$$\mathcal{S} \text{ Lie-irr.} \implies \mathcal{S} \text{ irr. }.$$

To any q-difference equation corresponds a q-difference system by the usual trick (the converse is also true : this result is known as the cyclic vector lemma), so that we can speak of the Galois group of a q-difference equation. The irreducibility of a q-difference operator L in the above "Galoisian" sense is equivalent to the irreducibility of L as an operator (immediate consequence of the Tannakian duality).

It is convenient and useful to introduce the notion of *q-difference module* (see [19, 16]). A q-difference module is a module over the non-commutative algebra (quantum algebra) $\mathcal{D}_q = \mathbb{C}(z)\langle \sigma_q, \sigma_q^{-1} \rangle$ of non-commutative polynomials satisfying the relation $\sigma_q z = qz\sigma_q$. The q-difference module associated to the q-difference operator $L \in \mathcal{D}_q$ is $\mathcal{D}_q/\mathcal{D}_q L$. The irreducibility of L as an operator is equivalent to the simplicity of the corresponding q-difference module.

The counterpart of the following result for generalized hypergeometric differential equations is due to Katz (see [11]). The q-difference case is treated in ([]).

Proposition 1. *The generalized q-hypergeometric system $\mathcal{H}_q(\underline{a};\underline{b})$ with parameters $(\underline{a},\underline{b}) \in (\mathbb{C}^*)^r \times (\mathbb{C}^*)^s$ is irreducible if and only if for all i,j we have $a_i/b_j \notin q^{\mathbb{Z}}$. In this case, for all $(\underline{\alpha},\underline{\beta}) \in \mathbb{Z}^r \times \mathbb{Z}^s$ the q-difference modules associated to $\mathcal{H}_q(\underline{a};\underline{b})$ and to $\mathcal{H}_q(q^{\alpha_1}a_1,\cdots,q^{\alpha_r}a_r;q^{\beta_1}b_1,\cdots,q^{\beta_s}b_s)$ are isomorphic (hence, they have isomorphic Galois groups).*

4 Preliminaries on algebraic group theory

Recall that $\mathsf{Sl}_2(\mathbb{C})$ admits up to isomorphism exactly one irreducible complex linear representation in each dimensions. The unique (up to isomorphism) irreducible representation of degree 3 of $\mathsf{Sl}_2(\mathbb{C})$ is $\mathsf{Sym}^{\otimes 2}(\mathsf{Std})$, the symmetric square of the standard representation of $\mathsf{Sl}_2(\mathbb{C})$ of degree 2, and is explicitly given by the following formula :

$$\begin{aligned} \rho : \mathsf{Sl}_2(\mathbb{C}) &\rightarrow \mathsf{Gl}_3(\mathbb{C}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}. \end{aligned} \tag{6}$$

The image of this representation, denoted $\mathsf{PSl}_2(\mathbb{C})$, is an algebraic subgroup of $\mathsf{Gl}_3(\mathbb{C})$ isomorphic to $\mathsf{Sl}_2(\mathbb{C})/\{\pm I_2\}$. Of course, this notation is not usual but will be very convenient.

For the following result, we refer to [18] and to the references therein.

Proposition 2. *Let G be a connected semi-simple algebraic subgroup of $\mathsf{Sl}_3(\mathbb{C})$ acting irreducibly on $\mathbb{C}^3$. Then we have the following alternative :*

- *either $G = \mathsf{Sl}_3(\mathbb{C})$;*
- *or G is conjugate to $\mathsf{PSl}_2(\mathbb{C})$.*

The previous Proposition can be also seen as a special case of the prime recognition lemma due to O. Gabber (see [11]).

Let us now list some properties of $\mathsf{PSl}_2(\mathbb{C})$ which will be useful in the rest of the paper. We set $\mathsf{PGl}_2(\mathbb{C}) = \mathbb{C}^* \cdot \mathsf{PSl}_2(\mathbb{C})$.

Lemma 1. *The normalizer of $\mathsf{PSl}_2(\mathbb{C})$ in $\mathsf{Gl}_3(\mathbb{C})$ is $\mathsf{PGl}_2(\mathbb{C})$.*

Proof. See [18, 17]. □

Lemma 2. *Let us consider $M \in \mathsf{PSl}_2(\mathbb{C}) \setminus \{I_3\}$.*

(i) The set of eigenvalues of M is of the form $\{1,\alpha,\alpha^{-1}\}$ with $\alpha \in \mathbb{C}^*$;

(ii) M is semi-simple iff $\alpha \neq 1$; in this case, M is conjugated to a diagonal matrix by an element of $\mathsf{PSl}_2(\mathbb{C})$;

(iii) if M is not semi-simple (that is $\alpha = 1$) then M is conjugated to $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ by an element of $\mathsf{PSl}_2(\mathbb{C})$.

Proof. Let N be an element of $\mathsf{Sl}_2(\mathbb{C})$ such that $M = \rho(N)$. If N is semi-simple, it is conjugated to some $\mathrm{diag}(\lambda, \lambda^{-1})$. Hence $M = \rho(N)$ is conjugated to $\mathrm{diag}(\lambda^2, 1, \lambda^{-2})$ by an element of $\mathsf{PSl}_2(\mathbb{C})$. On the other hand, if N is not semi-simple, then N is conjugated to $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Consequently, $M = \rho(N)$ is conjugated to $\rho\left(\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ by an element of $\mathsf{PSl}_2(\mathbb{C})$. The Lemma follows easily. □

We will also need the following elementary results, the proofs of which are left to the reader. We denote by Perm the finite subgroup of $\mathsf{Gl}_3(\mathbb{C})$ of permutation matrices; $\mathsf{Perm}_{\mathbb{C}^*}$ denotes the subgroup of $\mathsf{Gl}_3(\mathbb{C})$ generated by Perm and by the invertible diagonal matrices.

Lemma 3. *Let us consider $M \in \mathsf{Gl}_3(\mathbb{C})$ with three distinct eigenvalues. Suppose that $P, P' \in \mathsf{Gl}_3(\mathbb{C})$ are such that $P^{-1}MP$ and $P'^{-1}MP'$ are diagonal, then there exists $\Sigma \in \mathsf{Perm}_{\mathbb{C}^*}$ such that $P' = P\Sigma$.*

Lemma 4. *The normalizer of $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ in $\mathsf{Gl}_3(\mathbb{C})$ is made of upper-triangular matrices.*

5 Galois groups of Lie-irreducible equations

In order to prove our main Theorem 1 we study the non-resonant Lie-irreducible equations in several steps corresponding to various logarithmic and non-logarithmic cases.

5.1 Non-resonant and non-logarithmic case

Notations. It will be convenient to use the following notations. Let $\underline{u}$ be a 3-uple of $(\mathbb{C}^*)^3$.

- for all $\lambda \in \mathbb{C}^*$, $\lambda\underline{u} = (\lambda u_1, \lambda u_2, \lambda u_3)$ and $\lambda/\underline{u} = (\lambda/u_1, \lambda/u_2, \lambda/u_3)$;
- $V(\underline{u}) = \begin{pmatrix} 1 & 1 & 1 \\ q/u_1 & q/u_2 & q/u_3 \\ (q/u_1)^2 & (q/u_2)^2 & (q/u_3)^2 \end{pmatrix}$;
- $\mathrm{diag}(\underline{u}) = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}$;
- $\underline{u}_{\check{1}} = (u_2, u_3)$, $\underline{u}_{\check{2}} = (u_1, u_3)$ and $\underline{u}_{\check{3}} = (u_1, u_2)$;
- $(\underline{u}; q)_\infty = \lim_{n \to +\infty} (\underline{u}; q)_n$;
- $a_i = u_i q^{\alpha_i}$ and $b_j = v_j q^{\beta_j}$ with $u_i, v_j \in \mathbb{U}$ and $\alpha_i, \beta_j \in \mathbb{R}$ (we choose a logarithm of q).

Hypotheses. In this section we suppose that :

$$\forall i \neq j, \quad a_i/a_j \notin q^{\mathbb{Z}} \text{ and } b_2/b_3, b_2, b_3 \notin q^{\mathbb{Z}}. \tag{7}$$

Basic objects. Let us give the explicit form of the matrices involved in the density Theorem 2.

Local solution at 0. We have :

$$A(\underline{a};\underline{b};0)=V(\underline{b})J^{(0)}(\underline{b})V(\underline{b})^{-1}$$

with $J^{(0)}(\underline{b})=\mathrm{diag}(q/\underline{b})$. Hence the system (3) is non-resonant and non-logarithmic at 0 (that is $A(\underline{a};\underline{b};0)$ is semi-simple; hence we do not have to use the q-logarithm ℓ_q for solving the q-difference system defined by $A(\underline{a};\underline{b};0)$). A fundamental system of solutions at 0 of (3) as described in section 2.1 is given by $Y^{(0)}(\underline{a};\underline{b};z)=F^{(0)}(\underline{a};\underline{b};z)e^{(0)}_{J^{(0)}(\underline{b})}(z)$ with :

$$F^{(0)}(\underline{a};\underline{b};z)=\begin{pmatrix} {}_3\phi_2\left(\underline{a};\underline{b};z\right) & {}_3\phi_2\left(\frac{q}{b_2}\underline{a};\frac{q}{b_2}\underline{b};z\right) & {}_3\phi_2\left(\frac{q}{b_3}\underline{a};\frac{q}{b_3}\underline{b};z\right)\\ {}_3\phi_2\left(\underline{a};\underline{b};qz\right) & \frac{q}{b_2}{}_3\phi_2\left(\frac{q}{b_2}\underline{a};\frac{q}{b_2}\underline{b};qz\right) & \frac{q}{b_3}{}_3\phi_2\left(\frac{q}{b_3}\underline{a};\frac{q}{b_3}\underline{b};qz\right)\\ {}_3\phi_2\left(\underline{a};\underline{b};q^2z\right) & \left(\frac{q}{b_2}\right)^2{}_3\phi_2\left(\frac{q}{b_2}\underline{a};\frac{q}{b_2}\underline{b};q^2z\right) & \left(\frac{q}{b_3}\right)^2{}_3\phi_2\left(\frac{q}{b_3}\underline{a};\frac{q}{b_3}\underline{b};q^2z\right)\end{pmatrix}.$$

Generators of the local Galois group at 0. We have two generators :

$$\mathrm{diag}(e^{2\pi i\underline{\beta}})\text{ and }\mathrm{diag}(\underline{v}).$$

Local solution at ∞. We have :

$$A(\underline{a};\underline{b};\infty)=V(q\underline{a})J^{(\infty)}(\underline{a})V(q\underline{a})^{-1}$$

with $J^{(\infty)}(\underline{a})=\mathrm{diag}(1/\underline{a})$. Hence the system is non-resonant and non-logarithmic at ∞ and, a fundamental system of solutions at ∞ of (3) as described in section 2.1 is given by $Y^{(\infty)}(\underline{a};\underline{b};z)=F^{(\infty)}(\underline{a};\underline{b};z)e^{(\infty)}_{J^{(\infty)}(\underline{a})}(z)$ with :

$$F^{(\infty)}(\underline{a};\underline{b};z)=\begin{pmatrix} {}_3\phi_2\left(a_1q/\underline{b};a_1q/\underline{a};\frac{qb_2b_3}{a_1a_2a_3}z^{-1}\right) & {}_3\phi_2\left(a_2q/\underline{b};a_2q/\underline{a};\frac{qb_2b_3}{a_1a_2a_3}z^{-1}\right) & {}_3\phi_2\left(a_3q/\underline{b};a_3q/\underline{a};\frac{qb_2b_3}{a_1a_2a_3}z^{-1}\right)\\ \frac{1}{a_1}{}_3\phi_2\left(a_1q/\underline{b};a_1q/\underline{a};\frac{b_2b_3}{a_1a_2a_3}z^{-1}\right) & \frac{1}{a_2}{}_3\phi_2\left(a_2q/\underline{b};a_2q/\underline{a};\frac{b_2b_3}{a_1a_2a_3}z^{-1}\right) & \frac{1}{a_3}{}_3\phi_2\left(a_3q/\underline{b};a_3q/\underline{a};\frac{b_2b_3}{a_1a_2a_3}z^{-1}\right)\\ \left(\frac{1}{a_1}\right)^2{}_3\phi_2\left(a_1q/\underline{b};a_1q/\underline{a};\frac{b_2b_3}{qa_1a_2a_3}z^{-1}\right) & \left(\frac{1}{a_2}\right)^2{}_3\phi_2\left(a_2q/\underline{b};a_2q/\underline{a};\frac{b_2b_3}{qa_1a_2a_3}z^{-1}\right) & \left(\frac{1}{a_3}\right)^2{}_3\phi_2\left(a_3q/\underline{b};a_3q/\underline{a};\frac{b_2b_3}{qa_1a_2a_3}z^{-1}\right)\end{pmatrix}.$$

Generators of the local Galois group at ∞. We have two generators :

$$\breve{P}(y_0)^{-1}\mathrm{diag}(e^{2\pi i\underline{\alpha}})\breve{P}(y_0)\text{ and }\breve{P}(y_0)^{-1}\mathrm{diag}(\underline{u})\breve{P}(y_0).$$

Birkhoff matrix. The Barnes-Mellin-Watson formula (see [10] p. 110) entails that :

$$P(z)=(e^{(\infty)}_{J^{(\infty)}(\underline{a})}(z))^{-1}\begin{pmatrix} \frac{(\underline{a}_{\check{1}},\underline{b}_{\check{1}}/a_1;q)_\infty}{(\underline{b}_{\check{1}},\underline{a}_{\check{1}}/a_1;q)_\infty}\frac{\theta_q(a_1z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_2}\underline{a}_{\check{1}},\underline{b}_{\check{2}}/a_1;q\right)_\infty}{\left(\frac{q}{b_2}\underline{b}_{\check{2}},\underline{a}_{\check{1}}/a_1;q\right)_\infty}\frac{\theta_q(\frac{qa_1}{b_2}z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_3}\underline{a}_{\check{1}},\underline{b}_{\check{3}}/a_1;q\right)_\infty}{\left(\frac{q}{b_3}\underline{b}_{\check{3}},\underline{a}_{\check{1}}/a_1;q\right)_\infty}\frac{\theta_q(\frac{qa_1}{b_3}z)}{\theta_q(z)}\\ \frac{(\underline{a}_{\check{2}},\underline{b}_{\check{1}}/a_2;q)_\infty}{(\underline{b}_{\check{1}},\underline{a}_{\check{2}}/a_2;q)_\infty}\frac{\theta_q(a_2z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_2}\underline{a}_{\check{2}},\underline{b}_{\check{2}}/a_2;q\right)_\infty}{\left(\frac{q}{b_2}\underline{b}_{\check{2}},\underline{a}_{\check{2}}/a_2;q\right)_\infty}\frac{\theta_q(\frac{qa_2}{b_2}z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_3}\underline{a}_{\check{2}},\underline{b}_{\check{3}}/a_2;q\right)_\infty}{\left(\frac{q}{b_3}\underline{b}_{\check{3}},\underline{a}_{\check{2}}/a_2;q\right)_\infty}\frac{\theta_q(\frac{qa_2}{b_3}z)}{\theta_q(z)}\\ \frac{(\underline{a}_{\check{3}},\underline{b}_{\check{1}}/a_3;q)_\infty}{(\underline{b}_{\check{1}},\underline{a}_{\check{3}}/a_3;q)_\infty}\frac{\theta_q(a_3z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_2}\underline{a}_{\check{3}},\underline{b}_{\check{2}}/a_3;q\right)_\infty}{\left(\frac{q}{b_2}\underline{b}_{\check{2}},\underline{a}_{\check{3}}/a_3;q\right)_\infty}\frac{\theta_q(\frac{qa_3}{b_2}z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_3}\underline{a}_{\check{3}},\underline{b}_{\check{3}}/a_3;q\right)_\infty}{\left(\frac{q}{b_3}\underline{b}_{\check{3}},\underline{a}_{\check{3}}/a_3;q\right)_\infty}\frac{\theta_q(\frac{qa_3}{b_3}z)}{\theta_q(z)}\end{pmatrix}e^{(0)}_{J^{(0)}(\underline{b})}(z).$$

Twisted Birkhoff matrix. We deduce that :

$$\breve{P}(z)=\mathrm{diag}((1/z)^{-\underline{\alpha}})\begin{pmatrix}
\frac{(\underline{a}_{\check{1}},\underline{b}_{\check{1}}/a_1;q)_\infty}{(\underline{b}_{\check{1}},\underline{a}_{\check{1}}/a_1;q)_\infty}\frac{\theta_q(a_1z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_2}\underline{a}_{\check{1}},\underline{b}_{\check{2}}/a_1;q\right)_\infty}{\left(\frac{q}{b_2}\underline{b}_{\check{2}},\underline{a}_{\check{1}}/a_1;q\right)_\infty}\frac{\theta_q(\frac{qa_1}{b_2}z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_3}\underline{a}_{\check{1}},\underline{b}_{\check{3}}/a_1;q\right)_\infty}{\left(\frac{q}{b_3}\underline{b}_{\check{3}},\underline{a}_{\check{1}}/a_1;q\right)_\infty}\frac{\theta_q(\frac{qa_1}{b_3}z)}{\theta_q(z)}\\
\frac{(\underline{a}_{\check{2}},\underline{b}_{\check{1}}/a_2;q)_\infty}{(\underline{b}_{\check{1}},\underline{a}_{\check{2}}/a_2;q)_\infty}\frac{\theta_q(a_2z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_2}\underline{a}_{\check{2}},\underline{b}_{\check{2}}/a_2;q\right)_\infty}{\left(\frac{q}{b_2}\underline{b}_{\check{2}},\underline{a}_{\check{2}}/a_2;q\right)_\infty}\frac{\theta_q(\frac{qa_2}{b_2}z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_3}\underline{a}_{\check{2}},\underline{b}_{\check{3}}/a_2;q\right)_\infty}{\left(\frac{q}{b_3}\underline{b}_{\check{3}},\underline{a}_{\check{2}}/a_2;q\right)_\infty}\frac{\theta_q(\frac{qa_2}{b_3}z)}{\theta_q(z)}\\
\frac{(\underline{a}_{\check{3}},\underline{b}_{\check{1}}/a_3;q)_\infty}{(\underline{b}_{\check{1}},\underline{a}_{\check{3}}/a_3;q)_\infty}\frac{\theta_q(a_3z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_2}\underline{a}_{\check{3}},\underline{b}_{\check{2}}/a_3;q\right)_\infty}{\left(\frac{q}{b_2}\underline{b}_{\check{2}},\underline{a}_{\check{3}}/a_3;q\right)_\infty}\frac{\theta_q(\frac{qa_3}{b_2}z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_3}\underline{a}_{\check{3}},\underline{b}_{\check{3}}/a_3;q\right)_\infty}{\left(\frac{q}{b_3}\underline{b}_{\check{3}},\underline{a}_{\check{3}}/a_3;q\right)_\infty}\frac{\theta_q(\frac{qa_3}{b_3}z)}{\theta_q(z)}
\end{pmatrix}\mathrm{diag}(z^{-\underline{\beta}}).$$

It will be convenient to write :

$$p_{i,j}=p_{i,j}(\underline{a},\underline{b})=\frac{\left(\frac{q}{b_j}\underline{a}_{\check{i}},\underline{b}_{\check{j}}/a_i;q\right)_\infty}{\left(\frac{q}{b_j}\underline{b}_{\check{j}},\underline{a}_{\check{i}}/a_i;q\right)_\infty}.$$

For later use, we first compute the determinant and the minors of $\breve{P}(z)$.

Proposition 3. *We have :*

(i) the minor $(i_1,i_2)\times(j_1,j_2)$ of $\breve{P}(z)$ is equal to :

$$\begin{aligned}\kappa_{(i_1,i_2)\times(j_1,j_2)} \quad = \quad & \frac{-q}{(q;q)_\infty^2}\frac{a_{i_2}}{b_{j_1}}\frac{\left(\frac{q}{b_{j_1}}a_{i_3},b_{j_3}/a_{i_1},\frac{q}{b_{j_2}}a_{i_3},b_{j_3}/a_{i_2};q\right)_\infty\theta_q(\frac{a_{i_1}}{a_{i_2}})\theta_q(\frac{b_{j_1}}{b_{j_2}})}{\left(\frac{q}{b_{j_1}}\underline{b}_{\check{j}_1},\underline{a}_{\check{i}_1}/a_{i_1},\frac{q}{b_{j_2}}\underline{b}_{\check{j}_2},\underline{a}_{\check{i}_2}/a_{i_2};q\right)_\infty}\cdot\\ & \cdot(1/z)^{-(\alpha_{i_1}+\alpha_{i_2})}z^{-(\beta_{j_1}+\beta_{j_2})}\frac{\theta_q(\frac{q^2a_{i_1}a_{i_2}}{b_{j_1}b_{j_2}}z)}{\theta_q(z)}.\end{aligned}\tag{8}$$

(ii) the determinant of $\breve{P}(z)$ is equal to :

$$\begin{aligned}\det(\breve{P}(z)) \quad = \quad & -q\frac{(1-q/b_2)(1-q/b_3)(1/b_2-1/b_3)}{(1/a_2-1/a_1)(1/a_3-1/a_1)(1/a_2-1/a_3)}\cdot\\ & \cdot(1/z)^{-(\alpha_1+\alpha_2+\alpha_3)}z^{-(\beta_1+\beta_2+\beta_3)}\frac{\theta_q(\frac{q^2a_1a_2a_3}{b_2b_3}z)}{\theta_q(z)}.\end{aligned}\tag{9}$$

Proof. (i) The minor $(i_1,i_2)\times(j_1,j_2)$ of $\breve{P}(z)$ is equal to :

$$\begin{aligned}&\frac{(1/z)^{-(\alpha_{i_1}+\alpha_{i_2})}z^{-(\beta_{j_1}+\beta_{j_2})}}{\left(\frac{q}{b_{j_1}}\underline{b}_{\check{j}_1},\underline{a}_{\check{i}_1}/a_{i_1},\frac{q}{b_{j_2}}\underline{b}_{\check{j}_2},\underline{a}_{\check{i}_2}/a_{i_2};q\right)_\infty}\cdot\\ &\quad\cdot\Bigg(\left(\frac{q}{b_{j_1}}\underline{a}_{\check{i}_1},\underline{b}_{\check{j}_1}/a_{i_1},\frac{q}{b_{j_2}}\underline{a}_{\check{i}_2},\underline{b}_{\check{j}_2}/a_{i_2};q\right)_\infty\frac{\theta_q(\frac{qa_{i_1}}{b_{j_1}}z)}{\theta_q(z)}\frac{\theta_q(\frac{qa_{i_2}}{b_{j_2}}z)}{\theta_q(z)}\\ &\qquad-\left(\frac{q}{b_{j_1}}\underline{a}_{\check{i}_2},\underline{b}_{\check{j}_1}/a_{i_2},\frac{q}{b_{j_2}}\underline{a}_{\check{i}_1},\underline{b}_{\check{j}_2}/a_{i_1};q\right)_\infty\frac{\theta_q(\frac{qa_{i_2}}{b_{j_1}}z)}{\theta_q(z)}\frac{\theta_q(\frac{qa_{i_1}}{b_{j_2}}z)}{\theta_q(z)}\Bigg).\end{aligned}$$

Remark that the function :

$$\begin{aligned} f(z) &= \left(\frac{q}{b_{j_1}}\underline{a}_{\check{i}_1}, \underline{b}_{\check{j}_1}/a_{i_1}, \frac{q}{b_{j_2}}\underline{a}_{\check{i}_2}, \underline{b}_{\check{j}_2}/a_{i_2}; q\right)_\infty \theta_q(\frac{qa_{i_1}}{b_{j_1}}z)\theta_q(\frac{qa_{i_2}}{b_{j_2}}z) \\ &\qquad - \left(\frac{q}{b_{j_1}}\underline{a}_{\check{i}_2}, \underline{b}_{\check{j}_1}/a_{i_2}, \frac{q}{b_{j_2}}\underline{a}_{\check{i}_1}, \underline{b}_{\check{j}_2}/a_{i_1}; q\right)_\infty \theta_q(\frac{qa_{i_2}}{b_{j_1}}z)\theta_q(\frac{qa_{i_1}}{b_{j_2}}z) \end{aligned}$$

vanishes for $z \in q^{\mathbb{Z}}$ and $z \in \frac{b_{j_1}b_{j_2}}{q^2 a_{i_1}a_{i_2}}q^{\mathbb{Z}}$. Moreover, both functions f/θ_q^2 and $\frac{\theta_q(\frac{q^2 a_{i_1}a_{i_2}}{b_{j_1}b_{j_2}}z)}{\theta_q(z)}$ satisfy the same linear homogeneous q-difference equation of order one. We deduce that there exists a constant $\kappa \in \mathbb{C}^*$ such that, for all $z \in \mathbb{C}^*$, $(f/\theta_q^2)(z) = \kappa\frac{\theta_q(\frac{q^2 a_{i_1}a_{i_2}}{b_{j_1}b_{j_2}}z)}{\theta_q(z)}$. We get the value of κ by evaluating the above equality at $z = \frac{b_{j_1}}{qa_{i_2}}$; we obtain :

$$\begin{aligned} \kappa &= \frac{\left(\frac{q}{b_{j_1}}\underline{a}_{\check{i}_1}, \underline{b}_{\check{j}_1}/a_{i_1}, \frac{q}{b_{j_2}}\underline{a}_{\check{i}_2}, \underline{b}_{\check{j}_2}/a_{i_2}; q\right)_\infty \theta_q(\frac{a_{i_1}}{a_{i_2}})\theta_q(\frac{b_{j_1}}{b_{j_2}})}{\theta_q(\frac{b_{j_1}}{qa_{i_2}})\theta_q(q\frac{a_{i_1}}{b_{j_2}})} \\ &= \frac{-q}{(q;q)_\infty^2}\frac{a_{i_2}}{b_{j_1}}\left(\frac{q}{b_{j_1}}a_{i_3}, b_{j_3}/a_{i_1}, \frac{q}{b_{j_2}}a_{i_3}, b_{j_3}/a_{i_2}; q\right)_\infty \theta_q(\frac{a_{i_1}}{a_{i_2}})\theta_q(\frac{b_{j_1}}{b_{j_2}}) \end{aligned}$$

Therefore the minor $(i_1, i_2) \times (j_1, j_2)$ of $\check{P}(z)$ is equal to :

$$\frac{-q}{(q;q)_\infty^2}\frac{a_{i_2}}{b_{j_1}}\frac{\left(\frac{q}{b_{j_1}}a_{i_3}, b_{j_3}/a_{i_1}, \frac{q}{b_{j_2}}a_{i_3}, b_{j_3}/a_{i_2}; q\right)_\infty \theta_q(\frac{a_{i_1}}{a_{i_2}})\theta_q(\frac{b_{j_1}}{b_{j_2}})}{\left(\frac{q}{b_{j_1}}\underline{b}_{\check{j}_1}, \underline{a}_{\check{i}_1}/a_{i_1}, \frac{q}{b_{j_2}}\underline{b}_{\check{j}_2}, \underline{a}_{\check{i}_2}/a_{i_2}; q\right)_\infty} \cdot$$
$$\cdot (1/z)^{-(\alpha_{i_1}+\alpha_{i_2})} z^{-(\beta_{j_1}+\beta_{j_2})} \frac{\theta_q(\frac{q^2 a_{i_1}a_{i_2}}{b_{j_1}b_{j_2}}z)}{\theta_q(z)}$$

(ii) The following equality holds : $\det(\check{P}(z)) = (1/z)^{-(\alpha_1+\alpha_2+\alpha_3)} z^{-(\beta_1+\beta_2+\beta_3)} \frac{f^{(0)}(\underline{a};\underline{b};z)}{f^{(\infty)}(\underline{a};\underline{b};z)}$ with $f^{(0)}(\underline{a};\underline{b};z) = \det(F^{(0)}(\underline{a};\underline{b};z))$ and $f^{(\infty)}(\underline{a};\underline{b};z) = \det(F^{(\infty)}(\underline{a};\underline{b};z))$. From the functional equations verified by $f^{(0)}(\underline{a};\underline{b};z)$ and $f^{(\infty)}(\underline{a};\underline{b};z)$, and from the fact that these functions are respectively germs of holomorphic functions at 0 and at ∞, we deduce that $\frac{f^{(0)}(\underline{a};\underline{b};z)}{f^{(\infty)}(\underline{a};\underline{b};z)}$ is a meromorphic function over $\mathbb{C}^*$ whose poles are simples and located on $q^{\mathbb{Z}}$ and whose zeros are also simples and located on $\frac{b_2b_3}{q^2a_1a_3a_3}q^{\mathbb{Z}}$. Furthermore we have : $\frac{f^{(0)}(\underline{a};\underline{b};qz)}{f^{(\infty)}(\underline{a};\underline{b};qz)} = \frac{b_2b_3}{q^2a_1a_3a_3}\frac{f^{(0)}(\underline{a};\underline{b};z)}{f^{(\infty)}(\underline{a};\underline{b};z)}$. This ensures that there exists $\eta \in \mathbb{C}^*$ such that, for all $z \in \mathbb{C}^*$, $\frac{f^{(0)}(\underline{a};\underline{b};z)}{f^{(\infty)}(\underline{a};\underline{b};z)} = \eta\frac{\theta_q(\frac{q^2a_1a_3a_3}{b_2b_3}z)}{\theta_q(z)}$. We are going to determine the explicit value of η.

Expanding $\det(\check{P}(z))$ with respect to the first column, we get :

$$\begin{aligned}\det(\check{P}(z)) &= (1/z)^{-(\alpha_1+\alpha_2+\alpha_3)}z^{-(\beta_1+\beta_2+\beta_3)}\Bigg(\kappa_{(2,3)\times(2,3)}\frac{\theta_q(\frac{q^2a_2a_3}{b_2b_3}z)}{\theta_q(z)}\frac{\theta_q(a_1z)}{\theta_q(z)}\\ &\quad -\kappa_{(1,3)\times(2,3)}\frac{\theta_q(\frac{q^2a_1a_3}{b_2b_3})}{\theta_q(z)}\frac{\theta_q(a_2z)}{\theta_q(z)}\\ &\quad +\kappa_{(1,2)\times(2,3)}\frac{\theta_q(\frac{q^2a_1a_2}{b_2b_3})}{\theta_q(z)}\frac{\theta_q(a_3z)}{\theta_q(z)}\Bigg).\end{aligned}$$

We obtain the value of η by specializing the above equality at $z = 1/a_3$. Indeed a straightforward calculation shows that :

$$\begin{aligned}\eta\frac{\theta_q(\frac{q^2a_1a_2}{b_2b_3})}{\theta_q(1/a_3)} &= \frac{q}{(q;q)_\infty^6}\frac{1}{\left(\underline{b}_{\check{1}},\underline{a}_{\check{1}}/a_1,\frac{q}{b_2}\underline{b}_{\check{2}},\underline{a}_{\check{2}}/a_2,\frac{q}{b_3}\underline{b}_{\check{3}},\underline{a}_{\check{3}}/a_3;q\right)_\infty}\frac{a_2^2a_3}{b_2}\frac{\theta_q(\frac{b_2}{b_3})\theta_q(\frac{a_3}{a_2})\theta_q(\frac{a_1}{a_3})}{\theta_q(a_3)}\\ &\quad\cdot\left(\theta_q(a_2)\theta_q(\frac{b_2}{a_1})\theta_q(\frac{b_3}{a_1})\theta_q(\frac{q^2a_2}{b_2b_3})-\theta_q(a_1)\theta_q(\frac{b_2}{a_2})\theta_q(\frac{b_3}{a_2})\theta_q(\frac{q^2a_1}{b_2b_3})\right).\end{aligned}$$

Moreover :

$$\theta_q(a_2)\theta_q(\frac{b_2}{a_1})\theta_q(\frac{b_3}{a_1})\theta_q(\frac{q^2a_2}{b_2b_3})-\theta_q(a_1)\theta_q(\frac{b_2}{a_2})\theta_q(\frac{b_3}{a_2})\theta_q(\frac{q^2a_1}{b_2b_3}) = -a_2\theta_q(b_3)\theta_q(\frac{q^2a_1a_2}{b_2b_3})\theta_q(\frac{a_1}{a_2})\theta_q(b_2)$$

(use the above formula for f with a suitable choice of parameters). Hence, we have the formula :

$$\begin{aligned}\det(\check{P}(z)) &= -q\frac{(1-q/b_2)(1-q/b_3)(1/b_2-1/b_3)}{(1/a_2-1/a_1)(1/a_3-1/a_1)(1/a_2-1/a_3)}\\ &\quad\cdot(1/z)^{-(\alpha_1+\alpha_2+\alpha_3)}z^{-(\beta_1+\beta_2+\beta_3)}\frac{\theta_q(\frac{q^2a_1a_2a_3}{b_2b_3}z)}{\theta_q(z)}.\end{aligned}$$

□

Theorem 3. *Suppose that the q-hypergeometric equation $\mathcal{H}_q(\underline{a};\underline{b})$ is Lie-irreducible and has q-real parameters. Then $\mathsf{G}^{0,der} = \mathsf{Sl}_3(\mathbb{C})$. Moreover, we have :*

- *$\mathsf{G} = \mathsf{Gl}_3(\mathbb{C})$ if $\frac{a_1a_2a_3}{b_2b_3} \notin q^{\mathbb{Z}}$;*
- *$\mathsf{G} = \overline{\langle \mathsf{Sl}_3(\mathbb{C}), e^{2\pi i(\beta_2+\beta_3)}, v_1v_2\rangle}$ if $\frac{a_1a_2a_3}{b_2b_3} \in q^{\mathbb{Z}}$.*

Proof. Since G^0 acts irreducibly and faithfully on $\mathbb{C}^3$, G^0 is reductive. The general theory of algebraic groups entails that G^0 is generated by its center $Z(\mathsf{G}^0)$ together with its derived subgroup $\mathsf{G}^{0,der}$ which is semi-simple and that $Z(\mathsf{G}^0)$ acts as scalars. Hence, the connected semi-simple algebraic group $\mathsf{G}^{0,der} \subset \mathsf{Sl}_3(\mathbb{C})$ also acts irreducibly on $\mathbb{C}^3$. Therefore, Proposition 2 ensures that $\mathsf{G}^{0,der}$ is either conjugated to $\mathsf{PSl}_2(\mathbb{C})$ or equal to $\mathsf{Sl}_3(\mathbb{C})$.

Suppose that $\mathsf{G}^{0,der}$ is conjugated to $\mathsf{PSl}_2(\mathbb{C})$. Then, the Galois group G being a subgroup of the normalizer of $\mathsf{G}^{0,der}$, we deduce from Lemma 1 that G is a subgroup of some conjugate of $\mathsf{PGl}_2(\mathbb{C})$. Let $R \in \mathsf{Gl}_3(\mathbb{C})$ be such that $\mathsf{G} \subset R^{-1}\mathsf{PGl}_2(\mathbb{C})R$.

The non-resonance hypothesis implies that $\sharp\{e^{2\pi i\beta_1}, e^{2\pi i\beta_2}, e^{2\pi i\beta_3}\} = 3$ (resp. $\sharp\{e^{2\pi i\alpha_1}, e^{2\pi i\alpha_2}, e^{2\pi i\alpha_3}\} = 3$), that is, that the matrix $R\mathrm{diag}(e^{2\pi i\underline{\beta}})R^{-1}$ (resp. $R\breve{P}(z)^{-1}\mathrm{diag}(e^{2\pi i\underline{\alpha}})\breve{P}(z)R^{-1}$ for all $z \in \Omega$) has three distinct eigenvalues and hence, thanks to Lemma 2, is conjugated to some diagonal matrix (with three distinct diagonal entries) by some $\widetilde{R} \in \mathsf{PSl}_2(\mathbb{C})$ (resp. $\widetilde{R}_z \in \mathsf{PSl}_2(\mathbb{C})$). It follows from Lemma 3 that there exists $\Sigma \in \mathsf{Perm}_{\mathbb{C}^*}$ (resp. $\Sigma_z \in \mathsf{Perm}_{\mathbb{C}^*}$) such that $R = \widetilde{R}\Sigma$ (resp. $R\breve{P}(z)^{-1} = \widetilde{R}_z\Sigma_z$).

Consequently, for all $z \in \Omega$, $\Sigma_z\breve{P}(z)\Sigma^{-1} \in \mathsf{PSl}_2(\mathbb{C})$. Let us consider a set $\Omega' \subset \Omega$ with at least one accumulation point in $\mathbb{C}^*\backslash q^{\mathbb{Z}}$ such that there exists $\Sigma' \in \mathsf{Perm}$ such that, for all $z \in \Omega'$, there exists $(\lambda_{1,z}, \lambda_{2,z}, \lambda_{3,z}) \in (\mathbb{C}^*)^3$ with $\Sigma_z = \mathrm{diag}(\lambda_{1,z}, \lambda_{2,z}, \lambda_{3,z})\Sigma'$. In order to simplify the proof, we assume that $\Sigma = \Sigma' = I_3$: the reader will easily adapt the rest of the proof to the general case (permutation of the indices). In view of (6), we get that the entries of a matrix $M = (m_{i,j}) \in \mathsf{PSl}_2(\mathbb{C})$ satisfy $m_{1,2}^2 = 4m_{1,1}m_{1,3}$. Applying this formula to $M = \Sigma_z\breve{P}(z)\Sigma^{-1}$, we get a relation of the form :

$$\left(\frac{\left(\frac{q}{b_2}\underline{a}_{\check{1}}, \underline{b}_{\check{2}}/a_1; q\right)_\infty}{\left(\frac{q}{b_2}\underline{b}_{\check{2}}, \underline{a}_{\check{1}}/a_1; q\right)_\infty}\frac{\theta_q(\frac{qa_1}{b_2}z)}{\theta_q(z)}(1/z)^{-\alpha_1}z^{-\beta_2}\right)^2 = cst \cdot \frac{\left(\underline{a}_{\check{1}}, \underline{b}_{\check{1}}/a_1, \frac{q}{b_3}\underline{a}_{\check{1}}, \underline{b}_{\check{3}}/a_1; q\right)_\infty}{\left(\underline{b}_{\check{1}}, \underline{a}_{\check{1}}/a_1, \frac{q}{b_3}\underline{b}_{\check{3}}, \underline{a}_{\check{1}}/a_1; q\right)_\infty}\frac{\theta_q(a_1z)}{\theta_q(z)}\frac{\theta_q(\frac{qa_1}{b_3}z)}{\theta_q(z)}(1/z)^{-2\alpha_1}z^{-(\beta_1+\beta_3)}, \quad (10)$$

for all $z \in \Omega'$, where $cst \in \mathbb{C}^*$ does not depend on z. Since Ω' has an accumulation point in $\mathbb{C}^* \setminus q^{\mathbb{Z}}$, the principle of analytic continuation entails that (10) holds for all $z \in \widetilde{\mathbb{C}^*}$. Note that the Pochhammer symbols involved in the above functional equation are nonzero (if not, there would exist i, j such that $a_i/b_j \in q^{\mathbb{Z}}$: this is excluded since in this case the system is reducible). Now, the localization of the zeros of both sides of (10) leads us to a contradiction. Indeed, the left hand side vanishes on $\frac{qa_1}{b_2}q^{\mathbb{Z}}$ whereas the right hand side vanishes exactly on $a_1q^{\mathbb{Z}}$ and on $\frac{qa_1}{b_3}q^{\mathbb{Z}}$: this is a contradiction since the non-resonance hypothesis (7) implies that these three discrete q-logarithmic spirals are distinct.

Therefore $\mathsf{G}^{0,\mathsf{der}} = \mathsf{Sl}_3(\mathbb{C})$. The theorem follows from this and from formula (9). $\square$

5.2 $b_2 = b_3 \notin q^{\mathbb{Z}}$ and the system is non-resonant and non-logarithmic at ∞

Hypotheses. In this section we suppose that :

$$\forall i \neq j, \quad a_i/a_j \notin q^{\mathbb{Z}} \text{ and } b_2 = b_3 \notin q^{\mathbb{Z}}. \tag{11}$$

Basic objects.

It is easily seen that $A(\underline{a}; \underline{b}; 0)$ is conjugated to $J^{(0)}(\underline{b}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q/b_2 & 1 \\ 0 & 0 & q/b_2 \end{pmatrix}$.

Generators of the local Galois group at 0. We have three generators :

$$\mathrm{diag}(e^{2\pi i\underline{\beta}}),\ \mathrm{diag}(\underline{v}) \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b_2/q \\ 0 & 0 & 1 \end{pmatrix}.$$

In this section we do not need neither the explicit form of the local component at ∞ nor that of the twisted Birkhoff matrix in order to determine the derived group of the neutral component of the Galois group of a Lie-irreducible equation.

Theorem 4. *Let us suppose that the q-hypergeometric equation $\mathcal{H}_q(\underline{a};\underline{b})$ is Lie-irreducible. Then $\mathsf{G}^{0,der} = \mathsf{Sl}_3(\mathbb{C})$.*

Proof. Arguing as for the proof of Theorem 3, we get that $\mathsf{G}^{0,\mathsf{der}}$ is either conjugate to $\mathsf{PSl}_2(\mathbb{C})$, or equal to $\mathsf{Sl}_3(\mathbb{C})$ and that, in case that $\mathsf{G}^{0,\mathsf{der}}$ is conjugated to $\mathsf{PSl}_2(\mathbb{C})$, the Galois group G is a subgroup of some conjugate of $\mathsf{PGl}_2(\mathbb{C})$. Such an inclusion is impossible since $\mathsf{PGl}_2(\mathbb{C})$ contains exclusively semi-simple matrices or matrices with exactly one Jordan bloc, so that $\mathsf{PGl}_2(\mathbb{C})$ cannot contain a conjugate of $\begin{pmatrix}1&0&0\\0&1&b_2/q\\0&0&1\end{pmatrix}$. □

5.3 $b_2 = b_3 = q$ and the system is non-resonant and non-logarithmic at ∞

Notations. We set $\underline{q} = (q,q,q) \in (\mathbb{C}^*)^3$.

Hypotheses. In this section we suppose that :

$$\forall i \neq j, \quad a_i/a_j \notin q^{\mathbb{Z}} \text{ and } b_2 = b_3 = q. \tag{12}$$

Basic objects.

Local solution at 0. We have :

$$A(\underline{a};\underline{q};0) = \begin{pmatrix}1&-1&1\\1&0&0\\1&1&0\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&-1&1\\1&0&0\\1&1&0\end{pmatrix}^{-1}.$$

Consequently, we are in the non-resonant logarithmic case at 0. In order to get a fundamental system of solutions at 0 of (3) when $\underline{b} = \underline{q}$, we use a degeneration procedure.

We consider the limit as $\underline{b}$ tends to $\underline{q}$ of the following matrix valued function :

$$\begin{aligned}
&F^{(0)}(\underline{a};\underline{b};z)V(\underline{b})^{-1}\begin{pmatrix}1&-1&1\\1&0&0\\1&1&0\end{pmatrix}\\
= \quad & F^{(0)}(\underline{a};\underline{b};z)\begin{pmatrix}1 & \frac{1-q^2/(b_2b_3)}{(\frac{q}{b_2}-1)(\frac{q}{b_3}-1)} & \frac{q^2/(b_2b_3)}{(\frac{q}{b_2}-1)(\frac{q}{b_3}-1)}\\
0 & \frac{\frac{q}{b_3}-1}{(\frac{q}{b_3}-\frac{q}{b_2})(\frac{q}{b_2}-1)} & \frac{-\frac{q}{b_3}}{(\frac{q}{b_3}-\frac{q}{b_2})(\frac{q}{b_2}-1)}\\
0 & \frac{\frac{q}{b_2}-1}{(\frac{q}{b_2}-\frac{q}{b_3})(\frac{q}{b_3}-1)} & \frac{-\frac{q}{b_2}}{(\frac{q}{b_2}-\frac{q}{b_3})(\frac{q}{b_3}-1)}\end{pmatrix}.
\end{aligned}$$

An easy computation shows that this limit does exist and we denote it by $F^{(0)}(\underline{a};\underline{q};z)$. From (5) we deduce that $F^{(0)}(\underline{a};\underline{q};z)$ satisfies $F^{(0)}(\underline{a};\underline{q};qz)J^{(0)}(\underline{q}) = A(\underline{a};\underline{q};z)F^{(0)}(\underline{a};\underline{q};z)$ with $J^{(0)}(\underline{q}) = \begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$. Hence, the matrix $F^{(0)}(\underline{a};\underline{q};z)$ being invertible as a matrix in the field of meromorphic functions, the matrix valued function $Y^{(0)}(\underline{a};\underline{q};z) = F^{(0)}(\underline{a};\underline{q};z)e^{(0)}_{J^{(0)}(\underline{q})}(z)$ is a fundamental system of solutions of the q-hypergeometric equation with $\underline{b} = \underline{q}$. Recall that

$$e^{(0)}_{J^{(0)}(\underline{q})}(z) = \begin{pmatrix} 1 & \ell_q(z) & \frac{\ell_q(z)(\ell_q(z)-1)}{2} \\ 0 & 1 & \ell_q(z) \\ 0 & 0 & 1 \end{pmatrix}.$$

Generators of the local Galois group at 0. We have the following generator :

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Local solution at ∞. The situation is the same than in section 5.1. We are in the non-resonant and non-logarithmic case at ∞ and a fundamental system of solutions at ∞ of (3) as described in section 2.1 is given by $Y^{(\infty)}(\underline{a};\underline{q};z) = F^{(\infty)}(\underline{a};\underline{q};z)e^{(\infty)}_{J^{(\infty)}(\underline{a})}(z)$ with $J^{(\infty)}(\underline{a}) = \mathrm{diag}(1/\underline{a})$.

Generators of the local Galois group at ∞. We have two generators :

$$\breve{P}(y_0)^{-1}\mathrm{diag}(e^{2\pi i\underline{\alpha}})\breve{P}(y_0) \text{ and } \breve{P}(y_0)^{-1}\mathrm{diag}(\underline{u})\breve{P}(y_0).$$

Connection matrix. The connection matrix is the limit as $\underline{b}$ tends to $\underline{q}$ of :

$$(e^{(\infty)}_{J^{(\infty)}(\underline{a})}(z))^{-1}Qe^{(0)}_{J^{(0)}(\underline{q})}(z)$$

with :

$$Q = \begin{pmatrix} \frac{(\underline{a}_{\check{1}},\underline{b}_{\check{1}}/a_1;q)_\infty}{(\underline{b}_{\check{1}},\underline{a}_{\check{1}}/a_1;q)_\infty}\frac{\theta_q(a_1z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_2}\underline{a}_{\check{1}},\underline{b}_{\check{2}}/a_1;q\right)_\infty}{\left(\frac{q}{b_2}\underline{b}_{\check{2}},\underline{a}_{\check{1}}/a_1;q\right)_\infty}\frac{\theta_q(\frac{qa_1}{b_2}z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_3}\underline{a}_{\check{1}},\underline{b}_{\check{3}}/a_1;q\right)_\infty}{\left(\frac{q}{b_3}\underline{b}_{\check{3}},\underline{a}_{\check{1}}/a_1;q\right)_\infty}\frac{\theta_q(\frac{qa_1}{b_3}z)}{\theta_q(z)} \\ \frac{(\underline{a}_{\check{2}},\underline{b}_{\check{1}}/a_2;q)_\infty}{(\underline{b}_{\check{1}},\underline{a}_{\check{2}}/a_2;q)_\infty}\frac{\theta_q(a_2z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_2}\underline{a}_{\check{2}},\underline{b}_{\check{2}}/a_2;q\right)_\infty}{\left(\frac{q}{b_2}\underline{b}_{\check{2}},\underline{a}_{\check{2}}/a_2;q\right)_\infty}\frac{\theta_q(\frac{qa_2}{b_2}z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_3}\underline{a}_{\check{2}},\underline{b}_{\check{3}}/a_2;q\right)_\infty}{\left(\frac{q}{b_3}\underline{b}_{\check{3}},\underline{a}_{\check{2}}/a_2;q\right)_\infty}\frac{\theta_q(\frac{qa_2}{b_3}z)}{\theta_q(z)} \\ \frac{(\underline{a}_{\check{3}},\underline{b}_{\check{1}}/a_3;q)_\infty}{(\underline{b}_{\check{1}},\underline{a}_{\check{3}}/a_3;q)_\infty}\frac{\theta_q(a_3z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_2}\underline{a}_{\check{3}},\underline{b}_{\check{2}}/a_3;q\right)_\infty}{\left(\frac{q}{b_2}\underline{b}_{\check{2}},\underline{a}_{\check{3}}/a_3;q\right)_\infty}\frac{\theta_q(\frac{qa_3}{b_2}z)}{\theta_q(z)} & \frac{\left(\frac{q}{b_3}\underline{a}_{\check{3}},\underline{b}_{\check{3}}/a_3;q\right)_\infty}{\left(\frac{q}{b_3}\underline{b}_{\check{3}},\underline{a}_{\check{3}}/a_3;q\right)_\infty}\frac{\theta_q(\frac{qa_3}{b_3}z)}{\theta_q(z)} \end{pmatrix} \begin{pmatrix} 1 & \frac{1-q^2/(b_2b_3)}{(\frac{q}{b_2}-1)(\frac{q}{b_3}-1)} & \frac{q^2/(b_2b_3)}{(\frac{q}{b_2}-1)(\frac{q}{b_3}-1)} \\ 0 & \frac{\frac{q}{b_3}-1}{(\frac{q}{b_3}-\frac{q}{b_2})(\frac{q}{b_2}-1)} & \frac{-\frac{q}{b_3}}{(\frac{q}{b_3}-\frac{q}{b_2})(\frac{q}{b_2}-1)} \\ 0 & \frac{\frac{q}{b_2}-1}{(\frac{q}{b_2}-\frac{q}{b_3})(\frac{q}{b_3}-1)} & \frac{-\frac{q}{b_2}}{(\frac{q}{b_2}-\frac{q}{b_3})(\frac{q}{b_3}-1)} \end{pmatrix}$$

which has the following form :

$$(e^{(\infty)}_{J^{(\infty)}(\underline{a})}(z))^{-1} \begin{pmatrix} p_{1,1}(\underline{a})\frac{\theta_q(a_1z)}{\theta_q(z)} & A_1\frac{\theta_q(a_1z)}{\theta_q(z)} + p_{1,2}(\underline{a})a_1z\frac{\theta'_q(a_1z)}{\theta_q(z)} & B_2\frac{\theta_q(a_1z)}{\theta_q(z)} + C_1z\frac{\theta'_q(a_1z)}{\theta_q(z)} + \frac{a_1^2}{2}p_{1,3}(\underline{a})z^2\frac{\theta''_q(a_1z)}{\theta_q(z)} \\ p_{2,1}(\underline{a})\frac{\theta_q(a_2z)}{\theta_q(z)} & A_2\frac{\theta_q(a_2z)}{\theta_q(z)} + p_{2,2}(\underline{a})a_2z\frac{\theta'_q(a_2z)}{\theta_q(z)} & B_3\frac{\theta_q(a_2z)}{\theta_q(z)} + C_2z\frac{\theta'_q(a_2z)}{\theta_q(z)} + \frac{a_2^2}{2}p_{2,3}(\underline{a})z^2\frac{\theta''_q(a_2z)}{\theta_q(z)} \\ p_{3,1}(\underline{a})\frac{\theta_q(a_3z)}{\theta_q(z)} & A_3\frac{\theta_q(a_3z)}{\theta_q(z)} + p_{3,2}(\underline{a})a_3z\frac{\theta'_q(a_3z)}{\theta_q(z)} & B_3\frac{\theta_q(a_3z)}{\theta_q(z)} + C_3z\frac{\theta'_q(a_3z)}{\theta_q(z)} + \frac{a_3^2}{2}p_{3,3}(\underline{a})z^2\frac{\theta''_q(a_3z)}{\theta_q(z)} \end{pmatrix} e^{(0)}_{J^{(0)}(\underline{q})}(z)$$

where :

$$p_{i,j} = p_{i,j}(\underline{a}) = \frac{\left(\underline{a}_{\check{i}}, \underline{q}_{\check{j}}/a_i; q\right)_\infty}{\left(\underline{q}_{\check{j}}, \underline{a}_{\check{i}}/a_i; q\right)_\infty}.$$

Twisted connection matrix.

$$
\begin{aligned}
\breve{P}(z) \quad = \quad & \mathrm{diag}((1/z)^{-\underline{\alpha}}) \cdot \\
& \cdot \begin{pmatrix}
p_{1,1}(\underline{a})\frac{\theta_q(a_1z)}{\theta_q(z)} & A_1\frac{\theta_q(a_1z)}{\theta_q(z)} + p_{1,2}(\underline{a})a_1z\frac{\theta_q'(a_1z)}{\theta_q(z)} & B_2\frac{\theta_q(a_1z)}{\theta_q(z)} + C_1z\frac{\theta_q'(a_1z)}{\theta_q(z)} + \frac{a_1^2}{2}p_{1,3}(\underline{a})z^2\frac{\theta_q''(a_1z)}{\theta_q(z)} \\
p_{2,1}(\underline{a})\frac{\theta_q(a_2z)}{\theta_q(z)} & A_2\frac{\theta_q(a_2z)}{\theta_q(z)} + p_{2,2}(\underline{a})a_2z\frac{\theta_q'(a_2z)}{\theta_q(z)} & B_3\frac{\theta_q(a_2z)}{\theta_q(z)} + C_2z\frac{\theta_q'(a_2z)}{\theta_q(z)} + \frac{a_2^2}{2}p_{2,3}(\underline{a})z^2\frac{\theta_q''(a_2z)}{\theta_q(z)} \\
p_{3,1}(\underline{a})\frac{\theta_q(a_3z)}{\theta_q(z)} & A_3\frac{\theta_q(a_3z)}{\theta_q(z)} + p_{3,2}(\underline{a})a_3z\frac{\theta_q'(a_3z)}{\theta_q(z)} & B_3\frac{\theta_q(a_3z)}{\theta_q(z)} + C_3z\frac{\theta_q'(a_3z)}{\theta_q(z)} + \frac{a_3^2}{2}p_{3,3}(\underline{a})z^2\frac{\theta_q''(a_3z)}{\theta_q(z)}
\end{pmatrix} \cdot \\
& \cdot \begin{pmatrix} 1 & \ell_q(z) & \frac{\ell_q(z)(\ell_q(z)-1)}{2} \\ 0 & 1 & \ell_q(z) \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Theorem 5. *Suppose that the q-hypergeometric equation $\mathcal{H}_q(\underline{a};\underline{q})$ is Lie-irreducible and has q-real parameters. Then $\mathsf{G}^{0,der} = \mathsf{Sl}_3(\mathbb{C})$. Moreover, we have :*

- $\mathsf{G} = \mathsf{Gl}_3(\mathbb{C})$ *if* $a_1a_2a_3 \notin q^{\mathbb{Z}}$*;*
- $\mathsf{G} = \overline{\langle \mathsf{Sl}_3(\mathbb{C}), e^{2\pi i(\beta_2+\beta_3)}, v_1v_2\rangle}$ *if* $a_1a_2a_3 \in q^{\mathbb{Z}}$*.*

Proof. Arguing as for the proof of Theorem 3 we get that $\mathsf{G}^{0,der}$ is either conjugated to $\mathsf{PSl}_2(\mathbb{C})$, or equal to $\mathsf{Sl}_3(\mathbb{C})$.

Suppose that $\mathsf{G}^{0,der}$ is conjugated to $\mathsf{PSl}_2(\mathbb{C})$: there exists $R \in \mathsf{Gl}_3(\mathbb{C})$ such that $\mathsf{G}^{0,der} = R^{-1}\mathsf{PSl}_2(\mathbb{C})R$. The Galois group G being a subgroup of the normalizer of $\mathsf{G}^{0,der}$, we deduce from Lemma 1 that $\mathsf{G} \subset R^{-1}\mathsf{PGl}_2(\mathbb{C})R$.

We have $\sharp\{e^{2\pi i\alpha_1}, e^{2\pi i\alpha_2}, e^{2\pi i\alpha_3}\} = 3$. Considering the generator of the local Galois group at 0 given above, we see that $M = R\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}R^{-1}$ belongs to $\mathsf{PGl}_2(\mathbb{C})$. Let T' be an invertible upper-triangular matrix such that $T'\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}T'^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. We have $M = RT'\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}(RT')^{-1} \in \mathsf{PSl}_2(\mathbb{C})$. On the other hand, M being a non-semi-simple element of $\mathsf{PGl}_2(\mathbb{C})$, we deduce from Lemma 2 that there exists $S \in \mathsf{PSl}_2(\mathbb{C})$ such that $M = S\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}S^{-1}$. Considering the two expressions of M given above and using Lemma 4, we see that there exists T'' an invertible upper-triangular matrix such that $RT' = ST''$. Therefore the matrix $T = T''T'^{-1}$ is an invertible upper-triangular matrix and satisfies $R = ST$.

Proceeding as in the proof of Theorem 3, we see that, for all $z \in \Omega$, there exist $\Sigma_z \in \mathsf{Perm}_{\mathbb{C}^*}$ such that $\Sigma_z\breve{P}(z)R^{-1} \in \mathsf{PSl}_2(\mathbb{C})$.

We conclude that, for all $z \in \Omega$, $\Sigma_z\breve{P}(z)T^{-1} \in \mathsf{PSl}_2(\mathbb{C})$.

Using the algebraic relations verified by the entries of the elements of $\mathsf{PSl}_2(\mathbb{C})$, and arguing as for the proof of Theorem 3, we get that for some $i \in \{1,2,3\}$, a functional equation of the following form holds on $\mathbb{C}^*$:

$$\left(* \frac{\theta_q(a_i z)}{\theta_q(z)} + p_{i,2}(\underline{a}) a_i z \frac{\theta'_q(a_i z)}{\theta_q(z)} \right)^2 = K \frac{\theta_q(a_i z)}{\theta_q(z)} \left(* \frac{\theta_q(a_i z)}{\theta_q(z)} + * \frac{\theta'_q(a_i z)}{\theta_q(z)} + \frac{a_i^2}{2} p_{i,3}(\underline{a}) z^2 \frac{\theta''_q(a_i z)}{\theta_q(z)} \right)$$

for some constant $K \in \mathbb{C}^*$ and where each $*$ denotes a holomorphic function over $\mathbb{C}^* \setminus q^{\mathbb{R}}$. Now we get a contradiction : the right side of the above equality vanishes for $z \in \frac{1}{a_i} q^{\mathbb{Z}}$ but not the left hand side. Indeed, the vanishing of the right hand side on $z \in \frac{1}{a_i} q^{\mathbb{Z}}$ is clear; the non-vanishing of the left hand side on $z \in \frac{1}{a_i} q^{\mathbb{Z}}$ is a consequence of the fact that θ_q vanishes exactly to the order one on $q^{\mathbb{Z}}$ and that $p_{i,2}(\underline{a}) \neq 0$ (because the equation is irreducible). We get a contradiction.

Finally : $\mathsf{G}^{0,\mathsf{der}} = \mathsf{Sl}_3(\mathbb{C})$.

It is now easy to complete the proof using formula (9). □

5.4 $\underline{a}$ =(a,a,a) and $\underline{b}$=$\underline{q}$

Hypotheses. In this section we suppose that : $\underline{a} = (a,a,a)$ and $\underline{b} = \underline{q}$.

Basic objects.

Local solution at 0. We are in the non-resonant logarithmic case at 0 : the situation is the same than in the previous section. The matrix valued function $Y^{(0)}(a\underline{1};\underline{q};z) = F^{(0)}(a\underline{1};\underline{q};z) e^{(0)}_{J^{(0)}(\underline{q})}(z)$ is a fundamental system of solutions of the q-hypergeometric equation with $\underline{a} = a\underline{1}$ and $\underline{b} = \underline{q}$.

Generators of the local Galois group at 0. We have the following generator :

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Local solution at ∞. We have :

$$A(\underline{a};\underline{b};\infty) = \begin{pmatrix} 1/a^2 & -1/a & 1 \\ 1/a^3 & 0 & 0 \\ 1/a^4 & 1/a^3 & 0 \end{pmatrix} J^{(\infty)}(a\underline{1}) \begin{pmatrix} 1/a^2 & -1/a & 1 \\ 1/a^3 & 0 & 0 \\ 1/a^4 & 1/a^3 & 0 \end{pmatrix}^{-1}$$

with $J^{(\infty)}(a\underline{1}) = \begin{pmatrix} 1/a & 1 & 0 \\ 0 & 1/a & 1 \\ 0 & 0 & 1/a \end{pmatrix}$. We are in the non-resonant logarithmic case at ∞. A fundamental system of solutions at ∞ of (3) as described in section 2.1 is given by $Y^{(\infty)}(a\underline{1};\underline{q};z) =$

$F^{(\infty)}(a\underline{1};\underline{q};z)e^{(\infty)}_{J^{(\infty)}(\underline{a})}(z)$ where $F^{(\infty)}(a\underline{1};\underline{q};z)$ is the limit as $\underline{a}$ tends to $a\underline{1}$ of :

$$\begin{aligned} & F^{(\infty)}(\underline{a};\underline{q};z)V(q\underline{a})^{-1}\begin{pmatrix}1/a^2 & -1/a & 1\\ 1/a^3 & 0 & 0\\ 1/a^4 & 1/a^3 & 0\end{pmatrix} \\ = \;& F^{(\infty)}(\underline{a};\underline{q};z)\begin{pmatrix}a_1^2 & a_1^3/a_2 & a_1^3/a_2\\ 0 & -a_1^2/a_2+a_1^3/a_2^2 & -a_1^2/a_3+a_1^3/a_3^2\\ 0 & 1-2a_1/a_2+a_1^2/a_2^2 & 1-2a_1/a_3+a_1^2/a_3^2\end{pmatrix}^{-1}.\end{aligned}$$

Generators of the local Galois group at ∞. We have two generators :

$$e^{2\pi i\alpha}I_3,\ uI_3 \text{ and } \breve{P}(y_0)^{-1}\begin{pmatrix}1 & a & 0\\ 0 & 1 & a\\ 0 & 0 & 1\end{pmatrix}\breve{P}(y_0).$$

Connection matrix. The connection matrix is the limit as $\underline{a}$ tends to $a\underline{1}$ of :

$$\begin{aligned}&(e^{(\infty)}_{J^{(\infty)}(a\underline{1})}(z))^{-1}\begin{pmatrix}a_1^2 & a_1^3/a_2 & a_1^3/a_2\\ 0 & -a_1^2/a_2+a_1^3/a_2^2 & -a_1^2/a_3+a_1^3/a_3^2\\ 0 & 1-2a_1/a_2+a_1^2/a_2^2 & 1-2a_1/a_3+a_1^2/a_3^2\end{pmatrix}\cdot\\ &\cdot\begin{pmatrix}p_{1,1}(\underline{a})\frac{\theta_q(a_1z)}{\theta_q(z)} & A_1\frac{\theta_q(a_1z)}{\theta_q(z)}+p_{1,2}(\underline{a})a_1z\frac{\theta_q'(a_1z)}{\theta_q(z)} & B_2\frac{\theta_q(a_1z)}{\theta_q(z)}+C_1z\frac{\theta_q'(a_1z)}{\theta_q(z)}+\frac{a_1^2}{2}p_{1,3}(\underline{a})z^2\frac{\theta_q''(a_1z)}{\theta_q(z)}\\ p_{2,1}(\underline{a})\frac{\theta_q(a_2z)}{\theta_q(z)} & A_2\frac{\theta_q(a_2z)}{\theta_q(z)}+p_{2,2}(\underline{a})a_2z\frac{\theta_q'(a_2z)}{\theta_q(z)} & B_3\frac{\theta_q(a_2z)}{\theta_q(z)}+C_2z\frac{\theta_q'(a_2z)}{\theta_q(z)}+\frac{a_2^2}{2}p_{2,3}(\underline{a})z^2\frac{\theta_q''(a_2z)}{\theta_q(z)}\\ p_{3,1}(\underline{a})\frac{\theta_q(a_3z)}{\theta_q(z)} & A_3\frac{\theta_q(a_3z)}{\theta_q(z)}+p_{3,2}(\underline{a})a_3z\frac{\theta_q'(a_3z)}{\theta_q(z)} & B_3\frac{\theta_q(a_3z)}{\theta_q(z)}+C_3z\frac{\theta_q'(a_3z)}{\theta_q(z)}+\frac{a_3^2}{2}p_{3,3}(\underline{a})z^2\frac{\theta_q''(a_3z)}{\theta_q(z)}\end{pmatrix}e^{(0)}_{J^{(0)}(\underline{q})}(z)\end{aligned}$$

which has the following form :

$$(e^{(\infty)}_{J^{(\infty)}(a\underline{1})}(z))^{-1}\begin{pmatrix}* & * & *\\ * & * & *\\ \frac{\theta_q(a_1)^2}{(q;q)_\infty^4}\frac{\theta_q(a_1z)}{\theta_q(z)} & *\frac{\theta_q(a_1z)}{\theta_q(z)}+\frac{\theta_q(a_1)^2}{(q;q)_\infty^4}a_1z\frac{\theta_q'(a_1z)}{\theta_q(z)} & *\end{pmatrix}e^{(0)}_{J^{(0)}(\underline{q})}(z)$$

where each $*$ denotes an holomorphic function over $\mathbb{C}^*\setminus q^{\mathbb{R}}$.

Twisted connection matrix.

$$\begin{aligned}\breve{P}(z) \;=\; & (1/z)^{-\alpha}\begin{pmatrix}1 & a\ell_q(z) & a^2\frac{\ell_q(z)(\ell_q(z)-1)}{2}\\ 0 & 1 & a\ell_q(z)\\ 0 & 0 & 1\end{pmatrix}^{-1}\cdot\\ &\cdot\begin{pmatrix}* & * & *\\ * & * & *\\ \frac{\theta_q(a_1)^2}{(q;q)_\infty^4}\frac{\theta_q(a_1z)}{\theta_q(z)} & *\frac{\theta_q(a_1z)}{\theta_q(z)}+\frac{\theta_q(a_1)^2}{(q;q)_\infty^4}a_1z\frac{\theta_q'(a_1z)}{\theta_q(z)} & *\end{pmatrix}\begin{pmatrix}1 & \ell_q(z) & \frac{\ell_q(z)(\ell_q(z)-1)}{2}\\ 0 & 1 & \ell_q(z)\\ 0 & 0 & 1\end{pmatrix}\end{aligned}$$

where each $*$ denotes an holomorphic function over $\mathbb{C}^*\setminus q^{\mathbb{R}}$.

Theorem 6. *Let us suppose that the q-hypergeometric equation $\mathcal{H}_q(a\underline{1};\underline{q})$ is Lie-irreducible and has q-real parameters. Then $G^{0,der}=Sl_3(\mathbb{C})$. More precisely :*

- $G = Gl_3(\mathbb{C})$ *if* $a^3 \notin q^{\mathbb{Z}}$*;*

- $G = \overline{\langle Sl_3(\mathbb{C}), e^{2\pi i(\beta_2+\beta_3)}, v_1v_2\rangle}$ *if* $a^3 \in q^{\mathbb{Z}}$*.*

Proof. Arguing as for the proof of Theorem 3 we get that $\mathsf{G}^{0,\mathsf{der}}$ is either conjugated to $\mathsf{PSl}_2(\mathbb{C})$ or equal to $\mathsf{Sl}_3(\mathbb{C})$.

Suppose that $\mathsf{G}^{0,\mathsf{der}}$ is conjugated to $\mathsf{PSl}_2(\mathbb{C})$: there exists $R \in \mathsf{Gl}_3(\mathbb{C})$ such that $\mathsf{G}^{0,\mathsf{der}} = R^{-1}\mathsf{PSl}_2(\mathbb{C})R$. The Galois group G being included in the normalizer of $\mathsf{G}^{0,\mathsf{der}}$, we deduce from Lemma 1 that $\mathsf{G} \subset R^{-1}\mathsf{PGl}_2(\mathbb{C})R$.

Arguing as for Theorem 5, we get an invertible upper-triangular matrix T such that $RT^{-1} \in \mathsf{PSl}_2(\mathbb{C})$.

For similar reasons, for all $z \in \Omega$ there exists T_z an invertible upper-triangular matrix such that $T_z\breve{P}(z)R^{-1} \in \mathsf{PSl}_2(\mathbb{C})$.

We deduce that $T_z\breve{P}(z)T^{-1} \in \mathsf{PSl}_2(\mathbb{C})$.

Combining this fact with the explicit form of the connection matrix given above, we get a functional equation of the form :

$$\frac{\theta_q(a_1)^2}{(q;q)_\infty^4}\theta_q(a_1z)* = (*\theta_q(a_1z) + \frac{\theta_q(a_1)^2}{(q;q)_\infty^4}a_1z\theta_q'(a_1z))^2$$

where each $*$ denotes some holomorphic function over $\mathbb{C}^* \setminus q^{\mathbb{R}}$. Now we get a contradiction : the left hand side of the above functional equation vanishes for $z = 1/a_1$ but this is not the case of the right hand side since $\theta_q(a_1) \neq 0$ (the equation is irreducible).

Finally : $\mathsf{G}^{0,\mathsf{der}} = \mathsf{Sl}_3(\mathbb{C})$. The theorem follows easily from (9).

□

5.5 The remaining non-resonant cases

We have treated the following non resonant cases :

(i) $\forall i \neq j, \quad a_i/a_j \notin q^{\mathbb{Z}}$ and $b_2/b_3, b_2, b_3 \notin q^{\mathbb{Z}}$;

(ii) $\forall i \neq j, \quad a_i/a_j \notin q^{\mathbb{Z}}$ and $b_2 = b_3 \notin q^{\mathbb{Z}}$;

(iii) $\forall i \neq j, \quad a_i/a_j \notin q^{\mathbb{Z}}$ and $b_2 = b_3 = q$;

(iv) $\underline{a} = (a, a, a)$ and $\underline{b} = \underline{q}$.

We have shown that in each cases (in the q-real case) the Galois group G has $\mathsf{G}^{0,\mathsf{der}} = \mathsf{Sl}_3(\mathbb{C})$. We claim that the same conclusion holds for every non-resonant Lie-irreducible generalized q-hypergeometric equation of order three. Indeed, the reader will easily adapt the proofs of the cases (i), (ii), (iii) or (iv) to the remaining non-treated non-resonant (q-real) cases. For instance :

- if $a_1 = a_2 = a_3$ and $b_2/b_3, b_2, b_3 \notin q^{\mathbb{Z}}$, we proceed as for the case (iii);
- if $a_1 = a_2 \notin q^{\mathbb{Z}}a_3$, we proceed as for the case (ii), etc.

5.6 Proof of the main theorem

Theorem 7. *Let G be the Galois group of a Lie-irreducible generalized q-hypergeometric equation $\mathcal{H}_q(\underline{a};\underline{b})$ of order three with q-real parameters. Then $\mathsf{G}^{0,der} = \mathsf{Sl}_3(\mathbb{C})$. More precisely :*

- $\mathsf{G} = \mathsf{Gl}_3(\mathbb{C})$ *if* $\frac{a_1 a_2 a_3}{b_2 b_3} \notin q^{\mathbb{Z}}$;
- $\mathsf{G} = \overline{\langle \mathsf{Sl}_3(\mathbb{C}), e^{2\pi i(\beta_2+\beta_3)}, v_1 v_2 \rangle}$ *if* $\frac{a_1 a_2 a_3}{b_2 b_3} \in q^{\mathbb{Z}}$.

Proof. We have already shown that any non-resonant Lie-irreducible generalized q-hypergeometric equation of order three with q-real parameters has a Galois group G such that $\mathsf{G}^{0,\mathsf{der}} = \mathsf{Sl}_3(\mathbb{C})$. The proof follows easily from this and from Proposition 1, which allows us to reduce the problem to the non-resonant cases. □

As a concluding remark, it would be interesting to understand what happens to the difference Galois groups of the Lie-irreducible generalized q-hypergeometric equations under consideration in the present paper as q tends to 1, or, more generally, as $|q|$ tends to 1.

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